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On Generalized Gaussian Quadratures for Exponentials and Their Applications

G. Beylkin[1] and L. Monzón[2]

Department of Applied Mathematics, University of Colorado at Boulder, 526 UCB, Boulder, Colorado 80309

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We introduce new families of Gaussian-type quadratures for weighted integrals of exponential functions and consider their applications to integration and interpolation of bandlimited functions.

We use a generalization of a representation theorem due to Carathéodory to derive these quadratures. For each positive measure, the quadratures are parameterized by eigenvalues of the Toeplitz matrix constructed from the trigonometric moments of the measure. For a given accurac

the fo m

$$c_k = \sum_{j=1}^{M} \rho_j e^{i\pi\theta_j k}, \tag{1.1}$$

fo $k = 1, 2, \ldots, N$ and $M \leq N$, he e $-1 < \theta_j \leq 1$ and $\rho_j > 0$. Ca ath odo ep e entation (1.1) ha been the fo ndation fo a n mbe of algo ithm fo pect al e timation; in pa tic la , [20] i kno n in elect ical enginee ing lite at e a the Pi a enko method. In thi pape e de elop a fa t algo ithm fo nding M, the pha e T015

A a method fo con t ucting gene ali ed Ga ian q ad at e , o e lt a e limited to integ al (ith a fai l a bit a mea e) in ol ing exponential . O algo ithm in ol e nding eigen al e and eigen ecto of a Toeplit mat ix con t cted f om t igonomet ic moment of the mea e and then comp ting the oot on the nit ci cle fo app op iate eigenpol nomial . In pa tic la , each eigenpol nomial ith di tinct oot gi e i e to an identit hich, fo mall eigen al e , p o ide ith a Ga ian-t pe q ad at e and al o ith a ep e entation of po iti e de nite He mitian Toeplit mat ice . In the e identitie the i e of the eigen al e dete mine the acc ac of the q ad at e fo m la.

It t n o t that in the ca e of the eight leading to PSWF, the node of the co e ponding Ga ian q ad at e a e e o (app op iatel caled to the inte al [−1, 1]) of di c ete PSWF co e ponding to mall eigen al e .

A an application, e e the ne q ad at.7(I)-4.9(n)-199.2gTj/F.7ng2(cted)cTj4o2a . 4.7(

The paper is organized as follows. We present a brief description of the Pisarenko method to obtain the classical Carathéodory representation and we derive the estimate (1.2) in Section 2. In Section 3 we discuss generalized Gaussian quadratures for weighted integrals and prove some of their properties for weights supported inside $[-1/2, 1/2]$. In Section 4 we introduce new families of Gaussian-type quadratures. We develop a fast algorithm in Section 5 to compute the nodes and weights of these quadratures. We solve the approximation problem (1.3)–(1.5) in Section 6 and use it in the next two sections to obtain quadratures and interpolating bases for bandlimited functions. We also discuss various examples to illustrate these results. Finally, conclusions are presented in Section 9.

2. CARATHÉODORY REPRESENTATION

Carathéodory representation solves the trigonometric moment problem and can be stated as follows (see [8, Chap. 4]).

THEOREM 2.1. *Given N complex numbers $\boldsymbol{c} = (c_1, c_2, \ldots, c_N)$, not all zero, there exist unique $M \leq N$, positive numbers $\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots, \rho_M)$, and distinct real numbers $\theta_1, \theta_2, \ldots, \theta_M$*

2.1. *Algorithm I: Method to Obtain M, $\boldsymbol{\theta}$, and $\boldsymbol{\rho}$*

(1) Gi en $\boldsymbol{c} = (c_1, c_2, \ldots, c_N)$, e extend the de nition of c_k to negati e k a $c_{-k} = \overline{c_k}$ and e de ne c_0 o that the $(N+1) \times (N+1)$ Toeplit mat ix $\boldsymbol{T}_N$ of element $(\boldsymbol{T}_N)_{kj} = c_{j-k}$, ha nonnegati e eigen al e and at lea t one eigen al e i eq al to e o.

(2) De ne M a the ank of $\boldsymbol{T}_N$. B con t ction, e ha e $M \le N$. We al o a that M i the *rank* of the ep e entation (2.1).

(3) Let $\boldsymbol{T}_M$ be the top left p incipal bmat ix of o de $M+1$ of $\boldsymbol{T}_N$. That i , the mat ix $\boldsymbol{T}_M$ ha element $(c_{j-k})_{0 \le k,j \le M}$. Find the eigen ecto $\boldsymbol{q}$ co e ponding to the e o eigen al e of $\boldsymbol{T}_M$.

(4) Con t ct the pol nomial (eigenpol nomial) ho e coef cient a e the ent ie of the eigen ecto $\boldsymbol{q}$. A ho n in [8, p. 58], the M oot of thi eigenpol nomial a e di tinct and ha e ab ol te al e 1. The pha e of the e oot a e the n mbe θ_j.

(5) Find the eight $\boldsymbol{\rho}$ b ol ing the Vande monde tem (2.1) fo $k = 1, \ldots, M$. The ill, in addition, ati f $\sum_k \rho_k = c_0$.

Remark 2.1. With the exten ion of the eq ence c_k, (2.1) i alid fo $|k| \le N$. If $\boldsymbol{q} = (q_0, \ldots, q_M)$ i the eigen ecto obtained in pa t (3) of Algo ithm 2.1, then

$$\sum_{k=0}^{M} c_{k+s} q_k = 0, \tag{2.2}$$

fo all s, $-N \le s \le 0$. In othe o d , e ha e fo nd an o de -M ec ence elation fo the o iginal eq ence $\{c_k\}_{k=1}^N$.

Remark 2.2. In p actice, e a e inte e ted in ing Ca ath odo ep e entation if M i mall compa ed ith N, o mo e gene all , if mo t eight a e malle than the acc ac o ght. Ho e e , in ch ca e , $\boldsymbol{T}_N$ ha a la ge (n me ical) n ll b pace that ca e e e e n me ical p oblem in dete mining c_0, the ank M, and the eigen ecto $\boldsymbol{q}$.

Ne e thele , if the eq ence $\boldsymbol{c}$ i the t igonomet ic moment of an app op iate eight, e ill be able to modif the p e io method in o de to obtain the pha e θ_j in an ef cient manne . In thi etting, the pha e and eight in Ca ath odo ep e entation can be tho ght of a the node and eight of a Ga ian-t pe q ad at e fo eighted integ al . Once the pha e a e obtained, Theo em 2.2 a e that the comp tation of the eight i a ell-po ed p oblem. In Section 5.2 e p e ent a fa t algo ithm to obtain the eight b e al ating ce tain pol nomial at the node $e^{i\pi\theta_j}$.

Remark 2.3. Gi en an He mitian Toeplit mat ix $\boldsymbol{T}$, let con ide it malle t eigen al e $\lambda^{(N)}$

3. GENERALIZED GAUSSIAN QUADRATURES FOR EXPONENTIALS

3.1. Preliminaries: Chebyshev Systems

In this section we collect some definitions and results related to Chebyshev systems. We follow mostly Karlin and Studden [12] (see also [13]). Readers familiar with this topic may skip this section.

A family of $n+1$ real-valued functions $u_0, \ldots, u_n$ defined on an interval $I = [a, b]$ is a *Chebyshev system* (T-system) if any nontrivial linear combination

$$u(t) = \sum_{j=0}^{n} \alpha_j u_j(t) \tag{3.1}$$

has at most n zeros on the interval I. This property of a T-system can be viewed as a generalization of the same property for polynomials. Indeed, the family $\{1, t, t^2, \ldots, t^n\}$ provides the simplest example of a Chebyshev system.

Alternatively, a T-system over $[a, b]$ may be defined by the condition that the $n+1$ order determinants are nonvanishing,

$$\det \begin{pmatrix} u_0(t_0) & u_0(t_1) & \cdots & u_0(t_n) \\ u_1(t_0) & u_1(t_1) & \cdots & u_1(t_n) \\ \cdots & \cdots & \cdots & \cdots \\ u_n(t_0) & u_n(t_1) & \cdots & u_n(t_n) \end{pmatrix} \not\equiv 0, \tag{3.2}$$

whenever $a \le t_0 < t_1 < \cdots < t_n \le b$. Without loss of generality, the determinant can be assumed positive.

Let $u_0, \ldots, u_n$ be a T-system on the interval I. The moment space $\mathcal{M}_{n+1}$ with respect to $u_0, \ldots, u_n$ is defined as the set

$\mathcal{M}_{n+1} =$

THEOREM 3.5 [12, VI, Sec. 4]. *For the periodic T-system (3.5), a point*

In this section we start by using Carathéodory representation and Theorem 3.7 from the previous section, to construct two different Gaussian quadratures for integrals with weight w. These quadratures are exact for trigonometric polynomials of appropriate degree.

We then generalize these types of quadratures further and develop a new family of Gaussian-type quadratures. This family of quadrature formulas is parameterized by the eigenvalues of the Toeplitz matrix

$$\boldsymbol{T} = \{t_{l-k}\}_{0\le k,l\le N}. \tag{4.2}$$

Among these new quadrature formulas, only those corresponding to eigenvalues of small size are of practical interest. In fact, the size of the eigenvalue determines the error of the quadrature formula. To compute the weights and nodes of these quadratures, we develop a new algorithm which may be viewed as a (major) modification of Algorithm 2.1. The new algorithm is described in Section 5. The main results of this section are gathered in Theorem 4.1.

We start by using Theorem 3.7 to write

$$t_k = \sum_{j=1}^{N} \omega_j \,\mathrm{e}^{\mathrm{i}\pi\varphi_j k} + \omega_0(-1)^k, \qquad \text{for } |k|\le N, \tag{4.3}$$

for unique positive weights ω_j and phases φ_j in $(-1,1)$. Then, for any $A(z)=\sum_{|k|\le N} a_k z^k$ in $\mathcal{P}_N$, the space of Laurent polynomials of degree at most N, we have

$$\int_{-1}^{1} A(\mathrm{e}^{\mathrm{i}\pi\tau})w(\tau)\,d\tau = \sum_{|k|\le N} a_k t_k = \sum_{j=1}^{N} \omega_j A(\mathrm{e}^{\mathrm{i}\pi\varphi_j}) + \omega_0 A(-1), \tag{4.4}$$

for unique positive weights ω_j and nodes $\mathrm{e}^{\mathrm{i}\pi\varphi_j}$.

Alternatively, using Carathéodory representation (2.1) applied to the sequence $c_k = t_k$, $1\le k\le N$,

$$\begin{aligned}\int_{-1}^{1} A(\mathrm{e}^{\mathrm{i}\pi\tau})w(\tau)\,d\tau &= \sum_{j=1}^{M} \rho_j A(\mathrm{e}^{\mathrm{i}\pi\theta_j}) + (t_0-c_0)\frac{1}{2}\int_{-1}^{1} A(\mathrm{e}^{\mathrm{i}\pi\tau})\,d\tau \\ &= \sum_{j=1}^{M} \rho_j A(\mathrm{e}^{\mathrm{i}\pi\theta_j}) + \lambda^{(N)}\frac{1}{2}\int_{-1}^{1} A(\mathrm{e}^{\mathrm{i}\pi\tau})\,d\tau,\end{aligned} \tag{4.5}$$

where $c_0=\sum_{j=1}^{M}\rho_j$ and $\{\mathrm{e}^{\mathrm{i}\pi\theta_j}\}$ are the roots of the eigenpolynomial corresponding to the smallest eigenvalue $\lambda^{(N)}$ of $\boldsymbol{T}$.

Note that (4.5) is again valid for all $A(z)$ in $\mathcal{P}_N$ and that the positive weights ρ_j and phases θ_j in $(-1,1]$ are unique.

Thus, we have two different quadratures that may not coincide. However, by considering $w(\tau)$ supported inside $(-1/2,1/2)$, (3.12) implies that w_0 in (4.4) decreases exponentially fast with N and, since $\min w(\tau)=0$ for $|\tau|\le 1$, we have

$$\lim_{N\to\infty} \lambda^{(N)} = 0, \tag{4.6}$$

4.2. Gaussian-Type Quadratures on the Unit Circle

In this section we present the main results of the paper. We derive new Gaussian-type quadratures valid for any eigenvalue of the matrix $\boldsymbol{T}$ rather than just the smallest eigenvalue λ_N. These quadratures allow us to select the desired accuracy and thus, to construct accuracy-dependent families of quadratures.

The nodes of the quadratures in (4.5) are the roots of the eigenpolynomial corresponding to the least eigenvalue of $\boldsymbol{T}$ and, because of Carathéodory representation, we know that the roots are on the unit circle and that the weights are positive numbers. In our generalization, this standard property for the nodes and weights is no longer enforced. However, we will show that for nodes on the unit circle, the corresponding weights are real. Moreover, in all examples we have examined, for all small eigenvalues λ of $\boldsymbol{T}$, their negative weights are associated with the nodes outside the support of the weight and are comparable in size with λ. We believe this property to hold for a wide variety of weights.

We prove the following

THEOREM 4.1. *Assume that the eigenpolynomial $V^{(s)}(z)$ corresponding to the eigenvalue $\lambda^{(s)}$ of $\boldsymbol{T}$ has distinct, nonzero roots $\{\gamma_j\}_{j=1}^N$. Then there exist numbers $\{w_j\}_{j=1}^N$ such that*

(i) *For all Laurent polynomials $P(z)$ of degree at most N,*

$$\int_{-1}^{1} P(e^{i\pi t}) w(t)\, dt = \sum_{j=1}^{N} w_j P(\gamma_j) + \lambda^{(s)} \frac{1}{2} \int_{-1}^{1} P(e^{i\pi t})\, dt. \tag{4.12}$$

(ii) *For each root γ_k with $|\gamma_k| = 1$, the corresponding weight w_k is a real number and*

$$w_k = \int_{-1}^{1} |L_k^s(e^{i\pi t})|^2 w(t)\, dt - \lambda^{(s)} \frac{1}{2} \int_{-1}^{1} |L_k^s(e^{i\pi t})|^2\, dt, \tag{4.13}$$

where

$$L_k^s(z) = \frac{V^{(s)}(z)}{(V^{(s)})'(\gamma_k)(z - \gamma_k)} \tag{4.14}$$

is the Lagrange polynomial associated with the root γ_k.

(iii) *If $\lambda^{(s)}$ is a simple eigenvalue, then for $k = 1, \ldots, N$, the weight w_k is nonzero and*

$$\frac{1}{w_k} = \sum_{\substack{0 \le l \le N \\ l \ne s}} \frac{V^{(l)}(\gamma_k) V_*^{(l)}(\gamma_k)}{\lambda^{(l)} - \lambda^{(s)}}, \tag{4.15}$$

where $V_*^{(l)}(z) = \overline{V^{(l)}}(z^{-1})$ is the reciprocal polynomial of $V^{(l)}(z)$.

In particular, for each γ_k with $|\gamma_k| = 1$,

$$\frac{1}{w_k} = \sum_{\substack{0 \le l \le N \\ l \ne s}} \frac{|V^{(l)}(\gamma_k)|^2}{\lambda^{(l)} - \lambda^{(s)}}. \tag{4.16}$$

(iv) *If $\lambda^{(s)}$ is a simple eigenvalue and all roots γ_k are on the unit circle, then the set $\{w_k\}_{k=1}^N$ contains exactly s positive numbers and $N - s$ negative numbers.*

In particular, if $s = 0$ or $s = N$, then all w_k are negative or positive, respectively.

Remark 4.1. O app oach to obtain Ga ian q ad at e doe not e S ego pol nomial and i the efo e b tantiall diffe ent than the one in [11]. We b ie ex plain the app oach in [11]. Note that (4.9) and (4.10) ho that the pol nomial $\{V^{(k)}(z)\}$ a e o thogonal ith e pect to both the al inne p od ct fo t igonomet ic pol nomial and the eighted inne p od ct ith eight $w(t)$. We can al o con t ct S ego pol nomial $\{p_k(z)\}$ o thogonal ith e pect to $w(t)$ and ch that each $p_k(z)$ ha p eci e deg ee k [26]. Fo an k, the oot of $p_k(z)$ a e all in $|z| < 1$ [8].

S eg1 1

Fo γ_k o t ide the ppo t of the mea e, e ha e ob e ed (Fig . 2, 3, and 5 8) that

$$\sum_{l:\ \lambda^{(s)} > \lambda^{(l)}} |V^{(l)}(\gamma_k)|^2$$

i a con tant of mode ate i e.

Th , the econd te m in (4.17) i $O(1/\lambda^{(s)})$ and the eight i indeed negati e and o ghl of the i e of the eigen al e.

Remark 4.5. Fo the eight ith al e 1 in $(-1/2, 1/2)$ and 0 othe i e, the eigenpol nomial a e the di c ete PSWF. Fo the e f nction , e kno that all eigen al e a e imple and that all eigenpol nomial oot a e on the nit ci cle [23].

COROLLARY 4.1. *Under the assumptions of Theorem* 4.1*, it follows that the Toeplitz matrix* $\boldsymbol{T}$ *in* (4.2) *has the following representation as a sum of rank-1 Toeplitz matrices,*

$$(\boldsymbol{T} - \lambda^{(s)}\boldsymbol{I})_{kl} = \sum_{j=1}^{N} w_j \gamma_j^{l-k},$$

where $\lambda^{(s)}$*,* w_j*, and* γ_j *are as in (4.12).*

Thi co olla ho ld be compa ed ith Rema k 2.3 noting that, in the co olla , $\lambda^{(s)}$ i not nece a il the lea t eigen al e of $\boldsymbol{T}$. Fo an alte nati e de i ation ee [4].

Proof of Theorem 4.1. (1) Fo $\boldsymbol{x} = (x_0, \ldots, x_N) \in \mathbb{C}^{N+1}$, let de ne

$$A_{\boldsymbol{x}}(z) = \begin{cases} \displaystyle\sum_{l=-L}^{L} x_{l+L} z^l, & \text{if } N = 2 \ldots \\ & \end{cases}$$

<

(3) Let $P \in \mathcal{P}_N$; then $z^N P(z)$ is a polynomial of at most degree $2N$, and since $z^L V^{(s)}(z)$ is a polynomial of degree N, by Euclidean division, there exist polynomials $q(z)$ and $r(z)$ of degree at most N and $N-1$ such that

$$z^N P(z) = z^L V^{(s)}(z) q(z) + r(z).$$

Thus,

$$P(z) = V^{(s)}(z) Q(z) + R(z), \tag{4.19}$$

where $Q(z) \in \mathcal{P}_L$ and $R(z)$ has the form $R(z) = \sum_{k=1}^{N} r_k z^{-k}$ and hence

$$\int_{-1}^{1} R(e^{i\pi t})\,dt = 0.$$

Using the fact that $\{V^{(l)}\}_{l=0}^{N}$ is a basis of $\mathcal{P}_L$, we write

$$\overline{Q(e^{i\pi t})} = \sum_{l=0}^{N} d_l V^{(l)}(e^{i\pi t}),$$

where d_l are some complex coefficients.

Using (4.10) and (4.18), we multiply both sides of (4.19) by $w(t)$ and integrate to obtain

$$\int_{-1}^{1} P(e^{i\pi t}) w(t)\,dt = \quad {}^{N}$$

and thus, considering $k = j$, (4.15) follows. Note that we need $\lambda^{(s)}$ to be simple to guarantee $\lambda^{(l)} - \lambda^{(s)} \neq 0$, $l \neq s$ in (4.20).

If we view the left hand side of (4.20) as the entries A_{kj} of a matrix A and let B be the matrix of entries

$$B_{lk} = V^{(l)}(\gamma_k), \qquad \text{where } 0 \le l \le N,\ l \neq s, \text{ and } 1 \le k \le N, \tag{4.21}$$

we can prove (4.20) by showing that $BA = B$ and that B is nonsingular.

For the latter claim, we simply check that the columns of B are linearly independent. Indeed, let a_l, $l \neq s$, be constants such that

$$\sum_{l \neq s} a_l V^{(l)}(\gamma_k) = 0, \qquad \text{for } k = 1, \dots, N.$$

It follows that the polynomial $P(z) = \sum_{l \neq s} a_l V^{(l)}(z) \in \mathcal{P}_L$ has the $N = 2L$ distinct roots γ_k. Since P and $V^{(s)}$ have the same degree and the same N distinct roots, $P(z) = cV^{(s)}(z)$, for some constant c. By (4.9), $V^{(s)}(z)$ is orthogonal to all the other eigenpolynomials and so $a_l = 0$.

To show that $BA = B$, we substitute $P(z) = V^{(l)}(z)V^{(m)}(z)$ in (4.12) to obtain

$$\int_{-1}^{1} V^{(l)}(e^{i\pi t})$$

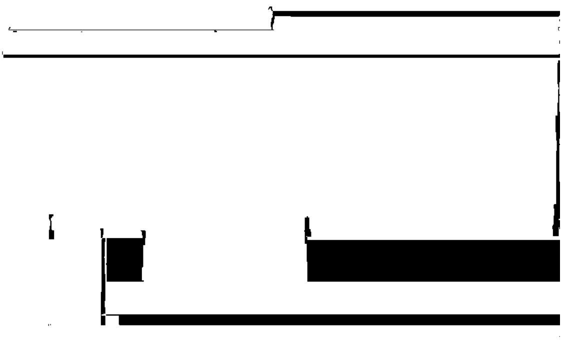

FIG. 2. Modified eigenpolynomial $e^{-i\pi t(N/2)}V^{(30)}(e^{i\pi t})$ on the interval $[-1,1]$, where $N = 97$ and $V^{(30)}(e^{i\pi t})$ is the eigenpolynomial corresponding to the eigenvalue $\lambda^{(30)}$ in Example 1. The phase factor $e^{-i\pi tN/2}$ is introduced to make this function real.

EXAMPLE 1. First we consider the weight

$$w(t) = \begin{cases} 1, & t \in [-a,a],\ a \leq 1/2, \\ 0, & \text{elsewhere}. \end{cases} \tag{4.24}$$

For this weight, the eigenpolynomial $V^{(l)}(e^{i\pi t})$ of the $N+1 \times N+1$ Toeplitz matrix $\boldsymbol{T}$ are the discrete PSWF [23]. Thus the eigenpolynomial $V^{(l)}(e^{i\pi t})$ has all of its zeros on the unit circle. Moreover, it has exactly l zeros for t in the interval $(-a,a)$ and N zeros for t in $[-1,1]$. In this example we have selected $N = 97$, $a = 1/6$, $c = 15\pi$. We then construct the matrix $\boldsymbol{T}$ and compute the eigenpolynomial corresponding to the eigenvalue

$$\lambda^{(30)} = 9.7730613638189163282 8 \cdot 10^{-16}. \tag{4.25}$$

The eigenpolynomial $V^{(30)}(e^{i\pi t})$ is shown in Figs. 2 and 3. Locations of the zeros on the unit circle are displayed in Fig. 4. We then use the quadrature formula corresponding to this eigenvalue and tabulate the weights in Table I. Note that the weights for nodes inside the interval $[-1/6, 1/6]$

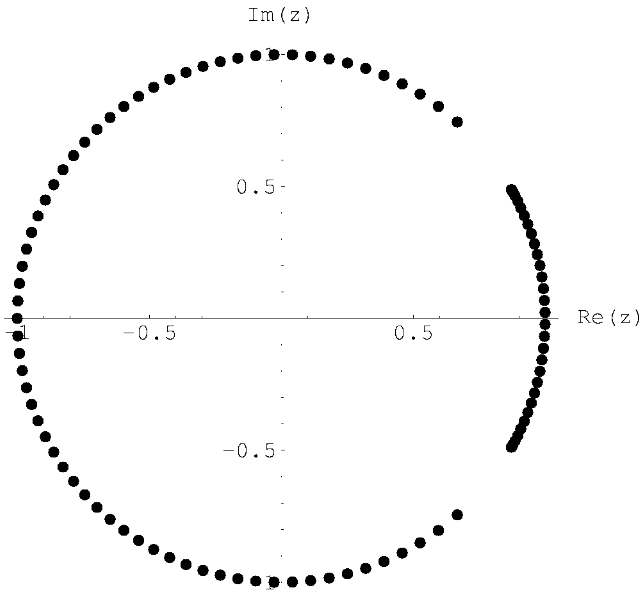

FIG. 4. Location of the zeros on the unit circle for the eigenpolynomial $V^{(30)}$ in Example 1.

TABLE I
Table of Weights for the Quadrature Formula with $\lambda^{(30)}$ in Example 1

#	Weight	#	Weight
1	$-1.0328 \cdot 10^{-17}$	50	0.04437549133235668283
2	$-1.0328 \cdot 10^{-17}$	51	0.04419611220330997984
3	$-1.0329 \cdot 10^{-17}$	52	0.04382960375644760677
⋮	⋮	53	0.04325984471286061543
		54	0.04246105337417774134
33	$-1.3518 \cdot 10^{-17}$	55	0.04139574827622469674
34	$-1.6030 \cdot 10^{-17}$	56	0.04001188663952018400
35	0.00580295532842819966	57	0.03823923547752508920
36	0.01310603337477264417	58	0.03598544514201341779
37	0.01959211245475268191	59	0.03313334531810570720
38	0.02506789313597245367	60	0.02954323947353217723
39	0.02954323947353217723	61	0.02506789313597245367
40	0.03313334531810570720	62	0.01959211245475268191
41	0.03598544514201341779	63	0.01310603337477264417
42	0.03823923547752508920	64	0.00580295532842819966
43	0.04001188663952018400	65	$-1.6030 \cdot 10^{-17}$
44	0.04139574827622469674	66	$-1.3518 \cdot 10^{-17}$
45	0.04246105337417774134		
46	0.04325984471286061543		
47	0.04382960375644760677		
48	0.04419611220330997984		
49	0.04437549133235668283		

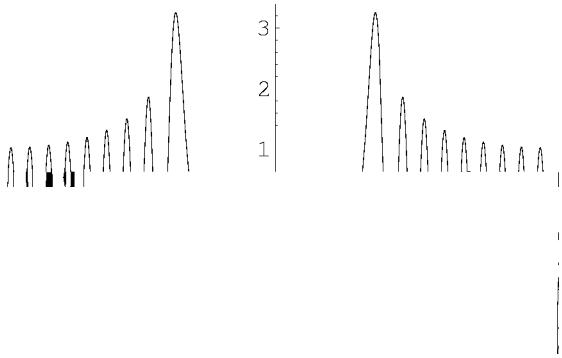

FIG. 5. Modified eigenpolynomial (see Fig. 2) on the interval $[-1, 1]$ corresponding to the eigenvalue $\lambda^{(28)}$ in Example 2.

EXAMPLE 2. We consider the weight

$$w(t) = \begin{cases} |t|/a, & t \in [-a, a],\ a \leq 1/2, \\ 0, & \text{elsewhere.} \end{cases} \tag{4.26}$$

In this example we have selected $N = 61$, $a = 1/4$, $c = 15\pi$. We then construct the matrix $\boldsymbol{T}$ and compute the eigenpolynomial corresponding to the eigenvalue

$$\lambda^{(28)} = 1.11598931688523706280 \cdot 10^{-14}. \tag{4.27}$$

The eigenpolynomial $V^{(28)}(e^{i\pi t})$ is shown in Figs. 5 and 6.

EXAMPLE 3. We consider a nonsymmetric weight

$$w(t) = \begin{cases} 1 + t/a, & t \in [-a, a],\ a \leq 1/2, \\ 0, & \text{elsewhere.} \end{cases} \tag{4.28}$$

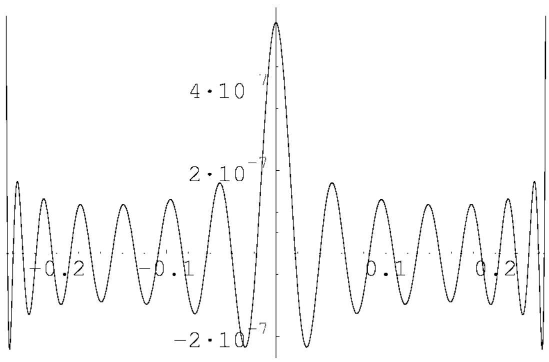

FIG. 6. The same function of Fig. 5 on the interval $[-1/4, 1/4]$.

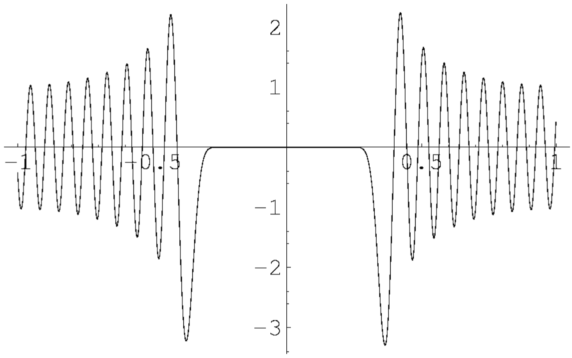

FIG. 7. Modi ed eigenpol nomial (ee Fig. 2) on the inte al $[-1, 1]$ co e ponding to the eigen al e $\lambda^{(28)}$ in Ex ample 3.

In thi ex ample e ha e elected $N = 61$, $a = 1/4$, $c = 15\pi$. We then con t ct the mat ix $\boldsymbol{T}$ and comp te the eigenpol nomial co e ponding to the eigen al e

$$\lambda^{(28)} = 4.68165338379692121389 \cdot 10^{-15}. \tag{4.29}$$

The eigenpol nomial $V^{(28)}(e^{i\pi t})$ i ho n in Fig . 7 and 8. Altho gh e do not ha e a p oof at the moment, it appea that the e i a cla of eight fo hich eigenpol nomial co e ponding to mall eigen al e mimic the beha io of the di c ete PSWF ith e pect to location of e o . In Ex ample 3 e kno that all e o a e on the nit ci cle d e to Theo em 4.2 and 4.3.

In Table II e ill t ate the pe fo mance of q ad at e fo diffe ent bandlimit c. Thi table ho ld be compa ed ith [29, Table 1]. The pe fo mance of both et of q ad at e i e imila . Yet the e q ad at e a e q ite diffe ent a can be een b compa ing Table III [29, Table 5]. Altho gh the acc ac i almo t identical, app ox imatel 10

TABLE II
Quadrature Performance for Varying Bandlimits

c	# of nodes	Maximum error
20	13	$1.2 \cdot 10^{-7}$
50	24	$1.1 \cdot 10^{-7}$
100	41	$1.6 \cdot 10^{-7}$
200	74	$1.8 \cdot 10^{-7}$
500	171	$1.4 \cdot 10^{-7}$
1000	331	$2.4 \cdot 10^{-7}$
2000	651	$1.2 \cdot 10^{-7}$
4000	1288	$3.7 \cdot 10^{-7}$

5. A NEW ALGORITHM FOR CARATHÉODORY REPRESENTATION

5.1. Algorithm 2

We now describe an algorithm for computing quadratures via a Carathéodory-type approach based on Theorem 4.1. It is easy to see that, although there are similarities with

TABLE III
Quadrature Nodes for Exponentials with Maximum Bandlimit $c = 50$

Node	Weight
−0.99041609489889	2.42209284787E-02
−0.95238829377394	5.04152570050E-02
−0.89243677566550	6.82109308489E-02
−0.81807124037876	7.96841731718E-02
−0.73438712699465	8.71710040243E-02
−0.64454148960251	9.22000859355E-02
−0.55050369342444	9.56668891250E-02
−0.45355265507507	9.80920675810E-02
−0.35456254990620	9.97843340729E-02
−0.25416536256280	1.00930070892E-01
−0.15284664158549	1.01641529848E-01
−0.05100535080412	1.01982696564E-01
0.05100535080412	1.01982696564E-01
0.15284664158549	1.01641529848E-01
0.25416536256280	1.00930070892E-01
0.35456254990620	9.97843340729E-02
0.45355265507507	9.80920675810E-02
0.55050369342444	9.56668891250E-02
0.64454148960251	9.22000859355E-02
0.73438712699465	8.71710040243E-02
0.81807124037876	7.96841731718E-02
0.89243677566550	6.82109308489E-02
0.95238829377394	5.04152570050E-02
0.99041609489889	2.42209284787E-02

Pi a enko’ method, the co e ponding algo ithm a e b tantiall diffe ent. We plan to add e implication fo ignal p oce ing in a epa ate pape .

(1) Gi en t_k, the t igonomet ic moment of a mea e, e con t ct the $(N+1) \times (N+1)$ Toeplit mat ix $\boldsymbol{T}_N$ ith element $(\boldsymbol{T}_N)_{kj} = t_{j-k}$. Thi mat ix i po iti e de nite and ha a la ge n mbe of mall eigen al e .

(2) Fo a gi en acc ac , e comp te the in e e of the Toeplit mat ix $\boldsymbol{T}_N - \boldsymbol{I}$. Fo a elf-adjoint Toeplit mat ix, it i f cient to ol e $(\boldsymbol{T}_N - \boldsymbol{I} pPo \pi \lambda prjV\ Q\lambda p\lambda \int eyr$

If we define

$$Q(z) = \prod_{k=1}^{M} (z - \gamma_k) = \sum_{k=0}^{M} q_k z^k, \tag{5.2}$$

then, for any polynomial P of degree at most $M - 1$,

$$\frac{P(z)}{Q(z)} = \sum_{r=1}^{M} \frac{P(\gamma_r)}{Q'(\gamma_r)(z - \gamma_r)}.$$

Thus, for $|z| < \min|\gamma_r|^{-1}$,

$$\frac{z^{M-1}}{z^M} \frac{P(z^{-1})}{Q(z^{-1})} = \sum_{r=1}^{M} \frac{P(\gamma_r)}{Q'(\gamma_r)} \sum_{k=0}^{+\infty} \gamma_r^k z^k = \sum_{k=0}^{+\infty} \left(\sum_{r=1}^{M} \frac{P(\gamma_r)}{Q'(\gamma_r)} \gamma_r^k \right) z^k. \tag{5.3}$$

Now choose P to be the unique polynomial with $P(\gamma_r) = \rho_r Q'$

This algorithm is equivalent to the following factorization of the inverse of the Vandermonde matrix in terms of a diagonal matrix, its transpose V^t, and a triangular Hankel matrix,

$$V^{-1} = \begin{pmatrix} \frac{1}{Q'(\gamma_1)} & \dots & 0 \\ & \ddots & \\ 0 & \dots & \frac{1}{Q'(\gamma_M)} \end{pmatrix} V^t \begin{pmatrix} q_1 & q_2 & \dots & q_M \\ q_2 & \dots & q_M & 0 \\ \vdots & & & \vdots \\ & & \dots & 0 \\ q_M & \dots & 0 & 0 \end{pmatrix}. \tag{5.5}$$

This description is a particular case of the inversion formulae for Lyness' Vandermonde [21] or closer to Vandermonde matrices [9, Corollary 2.1, p. 157]. We can state those results as (see [21, p. 548])

$$V^{-1} = \begin{pmatrix} x_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & x_M \end{pmatrix} V^t \begin{pmatrix} -y_2 & -y_3 & \dots & 1 \\ -y_3 & \dots & 1 & 0 \\ \vdots & & & \vdots \\ & & \dots & 0 \\ 1 & \dots & 0 & 0 \end{pmatrix},$$

where the vectors $\boldsymbol{x} = (x_1, \dots, x_M)^t$ and $\boldsymbol{y} = (y_1, \dots, y_M)^t$ are solutions of

$$\boldsymbol{V}\boldsymbol{x} = (0, \dots, 1)^t \qquad \text{and} \qquad \boldsymbol{V}^t\boldsymbol{y} = [\gamma_r^M]_{r=1}^M.$$

Since γ_r are the roots of $Q(z)$, we can take $\boldsymbol{y} = -(q_0, \dots, q_{M-1})^t$, and if $B(z) = z^M$ in (5.4), then $P(z) = 1$ and $\boldsymbol{x} = (1/Q'(\gamma_1), \dots, 1/Q'(\gamma_M))^t$.

Remark 5.1. For Algorithm 5.1, we just obtained the eigenvector $\boldsymbol{q}$ corresponding to an eigenvalue close to [illegible]. Thus, step (1) of the Vandermonde algorithm is already accomplished and step (2) can be performed using the FFT. Furthermore, the nodes γ_k belong to the unit circle and, via the unequally spaced fast Fourier transform, we have a fast algorithm to obtain the weights.

Remark 5.2. As an example, we use this approach to derive the solution of the Vandermonde system with nodes at $\gamma_r = e^{i2\pi(r-1)/M}$, $1 \le r \le M$. In this case, $Q(z) =$ $1 -$ [illegible]

Proof of Theorem 6.1. Let

$$u(y) = \int_{-1}^{1} \sigma(t) e^{i\pi t y}\, dt,$$

and, for each m, define the spline of order $2m-1$ interpolating $u(y)$ at the integers,

$$a(y) = \sum_k u(k) L_{2m-1}(y-k) = \int_{-1}^{1} \sigma(t) S_{2m-1}(y, e^{i\pi t})\, dt.$$

By (6.7),

$$|u(y) - a(y)| \le 3 \int_{-v}^{v} \sigma(t) |t|^{2m}\, dt \le 3 v^{2m} \|\sigma\|_1,$$

where $\|\sigma\|_1 = \int_{-1}^{1} \sigma(t)\, dt$. We choose m such that $3v^{2m}\|\sigma\|_1 < \varepsilon/4$.

On the other hand, for each N, Theorem 3.7 allows us to represent the moments $u(k)$, $|k| \le N$,

$$u(k) = \int_{-1}^{1} \sigma(t) e^{i\pi k t}\, dt = \sum_{j=1}^{N} w_j e^{i\pi \theta_j k} + w_0 (-1)^k, \tag{6.9}$$

where

$$w_0 \le \frac{4\|\sigma\|_1}{2 + (2+\sqrt{3})^N + (2-\sqrt{3})^N}. \tag{6.10}$$

Let

$$\tilde{u}(y) = \sum_{j=1}^{N} w_j e^{i\pi\theta_j y};$$

then $u(k) = \tilde{u}(k) + w_0(-1)^k$ for $|k| \le N$, and defining

$$\tilde{a}(y) = \sum_k \tilde{u}(k) L_{2m-1}(y-k) = \sum_{j=1}^{N} w_j S_{2m-1}(y, e^{i\pi\theta_j}),$$

(6.7) gives the estimate

$$|\tilde{u}(y) - \tilde{a}(y)| \le 3 \sum_{j=1}^{N} w_j |\theta_j|^{2m} \le 3v^{2m}(u(0) - w_0) \le 3v^{2m}\|\sigma\|_1 < \frac{\varepsilon}{4}.$$

We have shown that $u(y)$ is close to $a(y)$ and $\tilde{u}(y)$ is close to $\tilde{a}(y)$. To finish the proof, we need to show that $|a(y) - \tilde{a}(y)| < \varepsilon/2$, for $|y| \le dN + 1$. Now,

$$\begin{aligned} a(y) - \tilde{a}(y) &= \sum_{|k| \le N} w_0 (-1)^k L_{2m-1}(y-k) + \sum_{|k|>N} (u(k) - \tilde{u}(k)) L_{2m-1}(y-k) \\ &= w_0 S_{2m-1}(y, e^{i\pi}) + \sum_{|k|>N} (u(k) - \tilde{u}(k) - w_0(-1)^k) L_{2m-1}(y-k) \end{aligned}$$

and

$$\begin{aligned} |u(k) - \tilde{u}(k) - w_0(-1)^k| &\le |u(k)| + |\tilde{u}(k)| + w_0 \le \sum_{j=0}^{N} w_j + \sum_{j=1}^{N} w_j + w_0 \\ &\le 2u(0) = 2\|\sigma\|_1, \end{aligned}$$

where we used (6.9).

Since J_{2n} i an e en f nction, v e ha e

$$v(x) = \int_{-1}^{1} \tilde{w}(\tau) J_{2n}(cx\tau)\, d\tau. \tag{7.4}$$

U ing

$$J_{2n}(\xi) = \frac{(-1)^n}{\pi}$$

where

$$\tilde{v}_j = \sum_{k=1}^{M} w_k \psi_j(\theta_k), \tag{7.13}$$

and the nodes θ_k *and the weights* w_k *are the same as in* (1.4).

Fo la ge c, the pect m of F_c can be di ided into th ee g o p . The t g o p contain app ox imatel $2c/\pi$ eigen al e ith ab ol te al e e clo e to 1. The a e follo ed b o de $\log c$ eigen al e ho e ab ol te al e make an ex ponentiall fa t t an ition f om 1 to 0. The thi d g o p con i t of ex ponentiall deca ing eigen al e that a e e clo e to e o. Fo p eci e tatement ee [14, 24, 25, 29].

The efo e, it follo f om (7.12) that, fo the t $\approx 2c/\pi$ eigenf nction , the integ al in (7.11) a e ell app ox imated b the q ad at e in (7.13). To p o e (7.12), e (7.10), to ite

$$v_j - \tilde{v}_j = \frac{1}{\lambda_j} \int_{-1}^{1} \left(\int_{-1}^{1} w(\tau) e^{ic\tau t}\, d\tau - \sum_{k=1}^{M} w_k e^{ic\theta_k t} \right) \psi_j(t)\, dt. \tag{7.14}$$

Since $|t| \le 1$, e ha e

$$\left| \int_{-1}^{1} w(\tau) e^{ic\tau t}\, d\tau - \sum_{k=1}^{M} w_k e^{ic\theta_k t} \right| \le \quad , \tag{7.15}$$

and $\|\psi_j\|_2 = 1$ and implie TD0.0052 Tc(.14 41e012)Tj67 8Tj/(i)0.9(m Tm()8.6483 0 TD-94h5)T1 T

In con ide ing bandlimited f nction e ill e the PSWF (ee [15, 24], and a mo e ecent pa

$_c$ in (7.9) itheigen al

$_j$, j =

B etting

$$\alpha_l = w_l \sum_{j=0}^{M-1} \psi_j(b/c)\,\psi_j(t_l), \tag{8.18}$$

and ob e ing that $|\lambda_M| \approx$ and that $|\lambda_j| \square |\lambda_M|$ fo $j > M$, w e obtain (8.5) and (8.6). ■

We now con t ct t w o ef l ba e a linea combination of the f nction $\{e^{ict_l x}\}_{l=1}^{M}$. Fi t, let con ide the follow ing algeb aic eigen al e p oblem,

$$\sum_{l=1}^{M} w_l e^{ict_m t_l} \quad {}_j(t_l) = \eta_j \quad {}_j(t_m), \tag{8.19}$$

w he e t_l and w_l a e the ame a in (8.1). B ol ing (8.19), w e nd η_j and ${}_j(t_l)$. We then con ide f nction ${}_j$, $j = 1, \ldots, M$, de ned fo an x a

$$_j(x) = \frac{1}{\eta_j} \sum_{l=1}^{M} w_l e^{icxt_l} \quad {}_j(t_l). \tag{8.20}$$

The f nction ${}_j$ in (8.20) a e linea combination of the ex ponential {hen matrices, there exists an orthonormal basis of real eigenvectors. Thus, computed via (8.22), we assume q_l^j to be a real orthogonal matrix and then

$$\sum_{j=1}^{M} \sqrt{w_l}\, \psi_j(t_l)\, \psi_j(t_m) \sqrt{w_m} = \delta_{lm} \tag{8.23}$$

and

$$\sum_{l=1}^{M} \psi_j(t_l)\, w_l\, \psi_{j'}(t_l) = \delta_{jj'}. \tag{8.24}$$

We have

$$\int_{-1}^{1} \psi_j(t)\, \psi_{j'}(t)\, dt = \frac{1}{\eta_j \eta_{j'}} \sum_{l,l'=1}^{M} w_l w_{l'}\, \psi_j(t_l)\, \psi_{j'}(t_{l'}) \int_{-1}^{1} \mathrm{e}^{ict(t_l + t_{l'})}\, dt \tag{8.25}$$

and, from (8.1), we obtain

$$\left| \int_{-1}^{1} \psi_j(t)\, \psi_{j'}(t)\, dt - \frac{1}{\eta_j \eta_{j'}} \sum_{l,l'=1}^{M} w_l w_{l'}\, \psi_j(t_l)\, \psi_{j'}(t_{l'}) \sum_{k=1}^{M} w_k \mathrm{e}^{ict_k(t_l + t_{l'})} \right|$$

$$\leq \frac{2 \sum_{k=1}^{M} w_k}{|\eta_j|\,|\eta_{j'}|}. \tag{8.26}$$

Let us now construct interpolating bases as linear combinations of the exponentials $\{e^{icxt_l}\}_{l=1}^{n}$. We define functions R_k, $k = 1, \ldots, M$, as

$$R_k(x) = \sum_{l=1}^{M} r_{kl} e^{icxt_l}, \tag{8.27}$$

where

$$r_{kl} = \sum_{j=1}^{M} w_k \psi_j(t_k) \frac{1}{\eta_j} \psi_j(t_l) w_l = \sum_{j=1}^{M} \sqrt{w_k} q_k^j \frac{1}{\eta_j} q_l^j \sqrt{w_l}. \tag{8.28}$$

By direct evaluation in (8.19) and (8.23), we verify that functions R_k are interpolating,

$$R_k(t_m) = \delta_{km}. \tag{8.29}$$

Let us show that the integration of $R_k(t)e^{iat}$, where $|a| \le c$, yields a one-point quadrature rule of accuracy $O(\epsilon)$.

PROPOSITION 8.3. *For $|a| \le c$, let*

$$\beta_k = \int_{-1}^{1} R_k(t) e^{iat}\, dt - w_k e^{iat_k}. \tag{8.30}$$

Then we have

$$|\beta_k| \le \|\beta\|_2 \le \sqrt{M} \left(\frac{\max_{k=1,\ldots,M} |w_k|}{\min_{k=1,\ldots,M} |\eta_k|} \right) \epsilon^2, \tag{8.31}$$

where $\|\beta\|_2 = \sqrt{\sum_{k=1}^{M} |\beta_k|^2}$.

Proof. Using (8.27) and (8.29),

$$\sum_{l=1}^{M} r_{kl} \sum_{m=1}^{M} w_m e^{ict_m(t_l + a/c)} = \sum_{m=1}^{M} w_m R_k(t_m) e^{iat_m} = w_k e^{iat_k}, \tag{8.32}$$

and, therefore, β_k in (8.30) can be written as a matrix-vector multiplication $\beta_k = \sum_{l=1}^{M} r_{kl} s_l$, where

$$s_l = \int_{-1}^{1} e^{ict(t_l + a/c)}\, dt - \sum_{m=1}^{M} w_m e^{ict_m(t_l + a/c)}. \tag{8.33}$$

The inequality (8.31) is then obtained via the usual l^2-norm estimates, taking into account that the matrices q_k^j and q_l^j in (8.28) are orthogonal and that, for functions e^{iax}, where $|a| \le c$, (8.1) implies $|s_l| \le \epsilon^2$. ■

We have observed (via computation) that $\max_{k=1,\ldots,M} |w_k| = O(1)$ and $\min_{k=1,\ldots,M} |\eta_k| = O(\epsilon)$ in (8.31), thus resulting in $\|\beta\|_2 = O(\epsilon)$. Next we derive a weak estimate showing that the functions R_k are close to being an interpolating basis for band-limited exponentials.

PROPOSITION 8.4. *For every b, $|b| \le c$, let us consider the function*

$$ {}_b(t) = e^{ibt} - \sum_{k=1}^{M} e^{ibt_k} R_k(t). \tag{8.34}$$

Then, for every $|a| \le c$, we have

$$\left| \int_{-1}^{1} {}_b(t) e^{iat}\, dt \right| \le \left(1 + M \frac{\max_{k=1,\dots,M} |w_k|}{\min_{k=1,\dots,M} |\eta_k|} \right) {}^2. \tag{8.35}$$

Proof. Using (8.30), we have

$$\int_{-1}^{1} {}_b(t) e^{iat}\, dt = \int_{-1}^{1} e^{i(b+a)t}\, dt - \sum_{k=1}^{M} w_k e^{i(b+a)t_k} - \sum_{k=1}^{M} e^{ibt_k} {}_k, \tag{8.36}$$

where

$$ {}_k = \int_{-1}^{1} R_k(t) e^{iat}\, dt - w_k e^{iat_k}. \tag{8.37}$$

Applying (8.1), we obtain

$$\left| \int_{-1}^{1} {}_b(t) e^{iat}\, dt \right| \le {}^2 + \sqrt{M} \| \ \|_2. \tag{8.38}$$

The estimate (8.35) then follows from Proposition 8.3. ■

Remark 8.2. Using the functions R_k, $k = 1, \dots, M$, on a hierarchy of intervals, it is possible to construct a multiresolution basis (for a finite number of scales) similar to multiwavelet bases. We will consider such construction and its applications elsewhere.

8.1. Examples

For the weight

$$\omega(t) = \begin{cases} 1, & t \in [-a, a],\ a \le 1/2, \\ 0, & \text{otherwise}, \end{cases} \tag{8.39}$$

we construct a 30-node quadrature formula so that (8.1) is satisfied with ${}^2 \approx 10^{-15}$. We

2

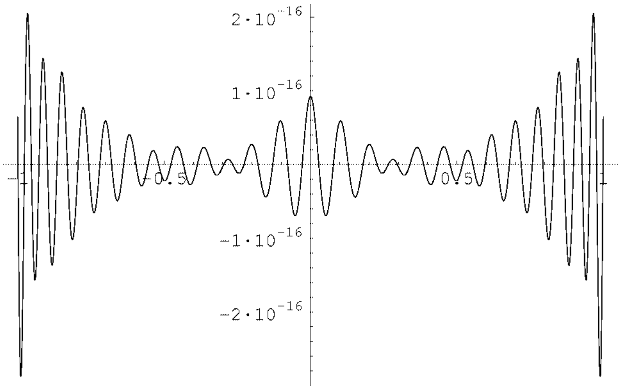

FIG. 9. E o in (8.1) fo Example 1.

he e P_9 i the Legend e pol nomial of deg ee 9. The e th ee f nction a e not pe iodic and e e

~

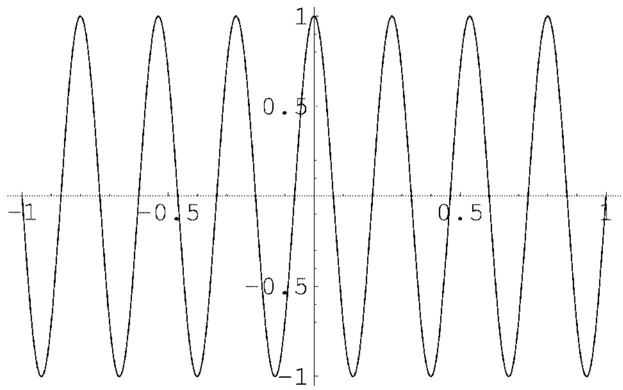

FIG. 11. F nction $g_1(t)$ on the inte al $[-1, 1]$.

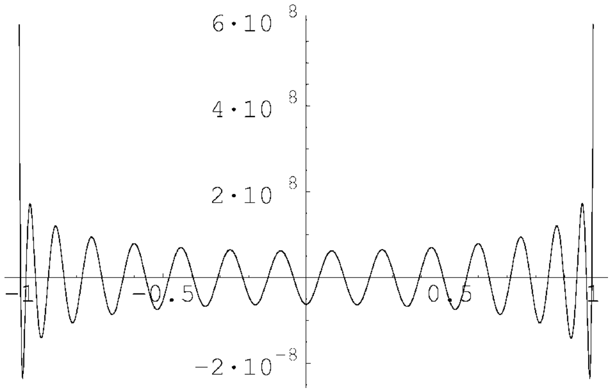

FIG. 12. Diffe ence $g_1(t) - \tilde{g}_1(t)$ on the inte al $[-1, 1]$.

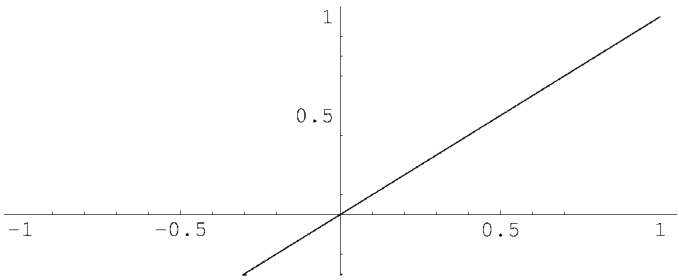

FIG. 13. F nction $g_2(t)$ on the inte al $[-1, 1]$.

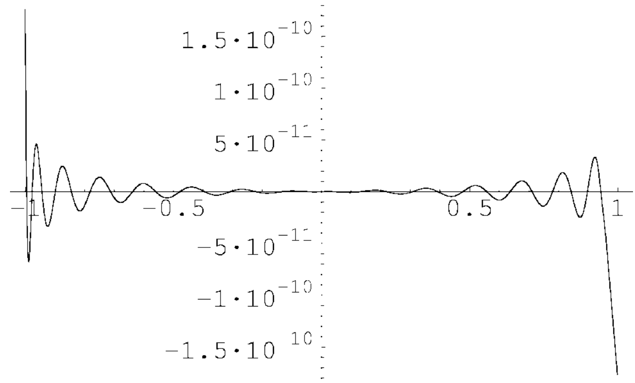

FIG. 14. Diffe ence $g_2(t) - \tilde{g}_2(t)$ on the inte al $[-1, 1]$.

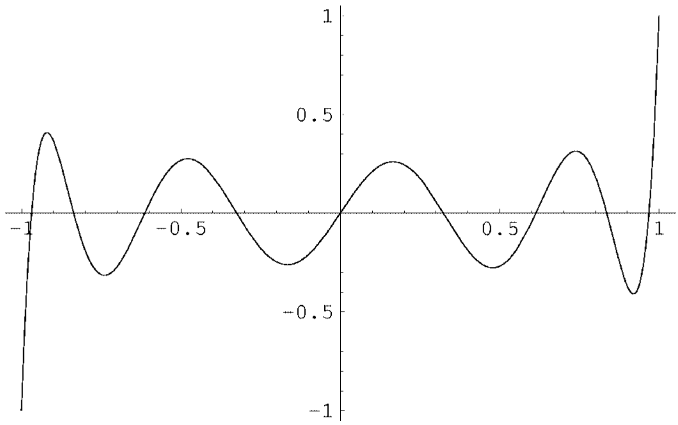

FIG. 15. F nction $g_3(t)$ on the inte al $[-1, 1]$.

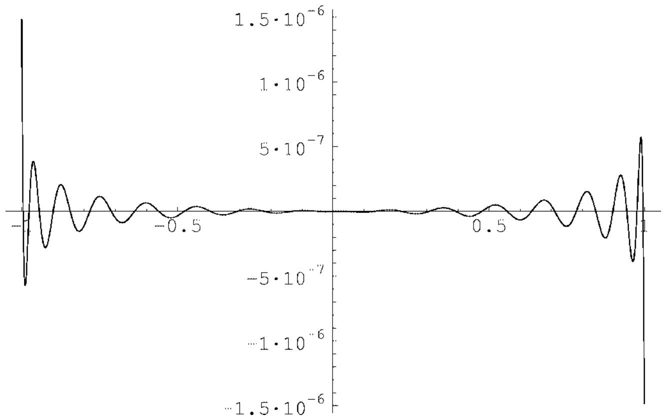

FIG. 16. Diffe ence $g_3(t) - \tilde{g}_3(t)$ on the inte al $[-1, 1]$.

exponential decay (see Fig. 1). For small eigenvalues, the quadratures are of practical interest.

The remarkable feature of these quadratures is that they have nodes outside the support of the measure and, as it turns out, the corresponding weights are negative and small, roughly of the size of the eigenvalue. The case corresponding to the smallest eigenvalue is equivalent to the classical Carathéodory representation.

As an application of the new quadratures, we show how to approximate and integrate real (exponentially) bandlimited functions. We also have constructed, using quadrature nodes and for a given precision, an interpolating basis for bandlimited functions on an interval.

In the paper we made a number of observations for which we do not have proofs. Let us finish by stating two unresolved issues. First, it is desirable to have tight uniform estimates for the L_∞-norm of the PSWF (with a fixed bandlimiting constant) or, ideally, for the eigenfunctions associated with more general weights. Second, we conjecture that in Theorem 4.1, it is not necessary to require distinct roots for the eigenpolynomial since it might be a consequence of the eigenvalues being simple. We have neither a proof nor a counterexample at this time.

APPENDIX: PROOF OF THEOREM 2.2

We use a technique that goes back to [2] (see [28, Theorem 7.3] and [19, Chapter 5] for more details) which involves the Fejér kernel,

$$F_L(x) = \sum_{|k|\le L} \left(1 - \frac{|k|}{L+1}\right) e^{i\pi k x} = \frac{\sin^2((L+1)\frac{\pi x}{2})}{(L+1)\sin^2\frac{\pi x}{2}}, \tag{A.1}$$

for real x.

We need the following result.

THEOREM A.1 [19, Theorem 8, Chapter 5]. *For $|k| \le N$, let*

$$c_k = \sum_{j=1}^{M} \rho_j z_j^k,$$

where $\rho_j \ge 0$ and $|z_j| = 1$. Then, for all L, $0 \le L \le N$,

$$(L+1)\|\boldsymbol{\rho}\|_2^2 \le c_0^2 + 2\sum_{k=1}^{L} |c_k|^2.$$

Proof. Let $a_k = 1 - |k|/L + 1$ be the coefficients of the Fejér kernel F_L and write $z_j = e^{i\pi\theta_j}$. Since $\rho_j \ge 0$ and $F_L(\theta) \ge 0$ for all θ,

$$\sum_{|k|\le L} a_k |c_k|^2 = \sum_{|k|\le L} a_k \sum_{j,l} \rho_j \rho_l \left(\frac{z_j}{z_l}\right)^k$$
$$= \sum_{j,l} \rho_j \rho_l F_L(\theta_j - \theta_l) \ge F_L(0) \sum_{j=1}^{M} \rho_j^2 = (L+1) \sum_{j=1}^{M} \rho_j^2.$$

The theorem follows because $a_0 = 1$ and $a_k \le 1$. ∎

Proof of Theorem 2.2. We use (2.1) to extend the definition of c_k as $c_{-k} = \overline{c_k}$ for $k = 1, \ldots, N$ and $c_0 = \sum_{j=1}^{M} \rho_j$. We then define the Toeplitz matrix $\boldsymbol{T}_N$, $(\boldsymbol{T}_N)_{kj} = (c_{j-k})_{0 \le k,j \le N}$, and the polynomial

$$Q(z) = \prod_{j=1}^{M} (z - \mathrm{e}^{\mathrm{i}\pi\theta_j}) = \sum_{k=0}^{M} q_k z^k.$$

Then $\boldsymbol{q} = (q_0, \ldots, q_M, 0, \ldots, 0)^t$

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